VI. *The Theory of Continuous Calculating Machines and of a Mechanism of this class on a New Principle.*

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There are an almost unlimited number of applications for a mechanism by means of which the velocity ratio between two rotating pieces could either be determined at any instant, or be made to vary in any required manner. Such a mechanism would enable two variable quantities to be dealt with numerically, for with it the operations of differentiation and integration could be mechanically performed.

For differentiation it would be necessary to cause two rotating bodies, which might be disks or rollers, to be turned at speeds which varied respectively according to the rates of change of the variable quantities, one quantity being a function of the other, when the velocity ratio, shown by a suitable index, would give their differential coefficient. This is evident, because velocity is simply the space passed over in a given time, and is, in the limit, proportional to the increment of this space. For integration, while one body is moved at a rate which changes with the independent variable, the velocity ratio of the two bodies (*i.e.*, the two rotating disks or rollers) must be made proportional to the dependent variable. The actual velocity of the second or driven rotating body then becomes a measure of the product of the latter into the rate of change of the former at the same instant. The motion of the driven body, as recorded for any period of time by a suitable index, is therefore a measure of the integration of their product for that time.

The primary or simple form of mechanism has but small value in comparison with the possibilities which a combination of such mechanisms seems to offer. The index, which shows the differential coefficient, or the driven disk which records the results of integration, can have respectively but one position or rate of motion at any instant. By employing, however, a suitable series or chain of such mechanisms, if need be of both kinds, the final index or the final disk could be made to either indicate or record, as the case might be, the result of any required number of conditions. It is difficult to say what limit there would be to the powers of the continuous calculator which such a combination would form.

The only hitherto known mechanism by which both the foregoing operations can be,

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in theory, effected appears to be that commonly referred to as the "disk and roller," with its modification, the disk-globe and cylinder-mechanism, or apparatus such as cones and belting which when analysed are all found to depend for their action upon the same kinematic principles. The action of the mechanism is simple enough, the disk, which is usually the piece to which motion is primarily imparted, drives the roller with a speed proportional both to its own angular velocity, and to the distance of the roller from its centre. Pieces of apparatus, such as cones and belting, apart from their unavoidable inaccuracy, are only suitable for the latter of the two above operations, viz., integration, and will therefore not be further alluded to.

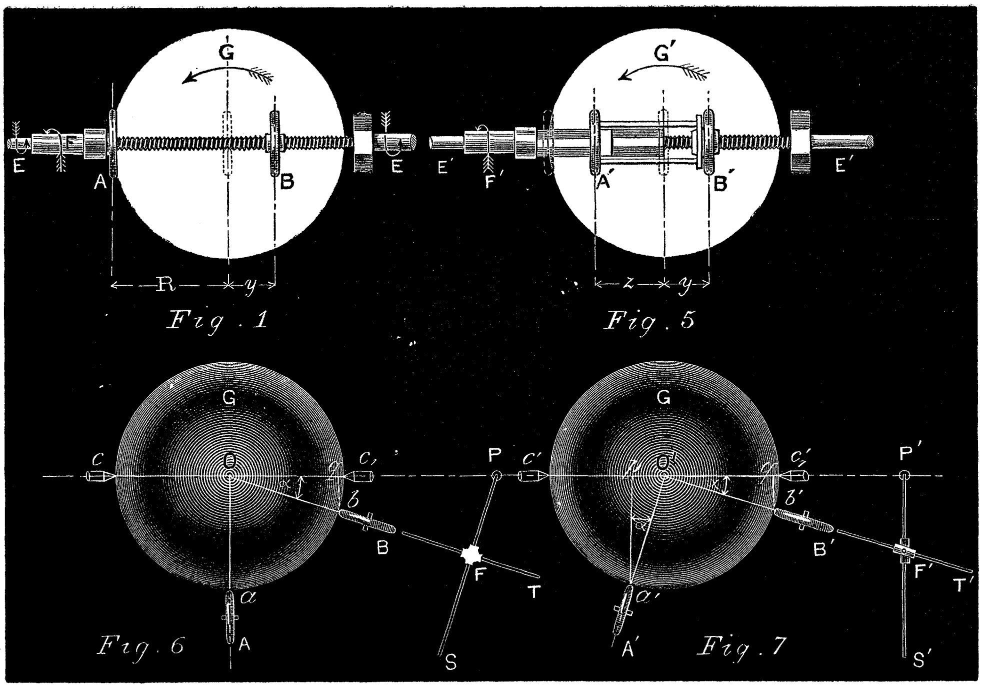
The disk and roller mechanism is chiefly known as a mechanical integrator. Its application for this purpose is due to PONCELET, who first suggested its use for ergometrical purposes as set forth in his 'Mécanique Industrielle,' and it was thus employed by General MORIN. In this case it was used to perform the operation $\int F ds$, where F was the force of traction measured by a spring, *ds* being the increment of space passed over at that instant by the cart. It was soon afterwards employed by Professor MOSELEY in a "constant steam engine indicator" designed for the use of a committee of the British Association.* In this instrument the disk was replaced by a cone, and F was the varying pressure of steam in the cylinder. Since then the disk and roller principle has been adopted by many inventors for purposes of integration in conjunction with dynamometers for measuring the power transmitted from a prime mover or to a machine. The cone and roller principle was employed by Professor MOSELEY in a calculating machine,† and by SANG and others as a Platometer or Planimeter. Messrs. ASHTON and STORY have revived the application of the disk and roller to a continuous Indicator, which should, however, really be called a steam-power *Integrator*. In the last-named instrument advantage is taken of the fact that as the steam pressure acts alternately on one side or the other of the piston, the roller may be made to take a corresponding position on either side of the centre of the disk, thus both the forward and backward stroke of the engine combine to effect the continuous rotation of the roller in one direction, and thereby integrate the total work done.

All the foregoing examples are applications of the disk and roller mechanism for purposes of integration. Differentiation may practically be effected by means of the disk and roller in the following manner:—In fig. 1 let G represent the disk, and B the movable roller. Instead of the roller B being fixed to its axis the latter is a screw (E E) of which B forms the nut. For convenience the disk is supposed to be driven from the hollow shaft F by means of the roller A, which is always at a constant distance from the centre of the disk, and may be therefore alluded to as having the same motion as the disk. Suppose this screw (EE) to be turned by one body whose rotary motion varies with the rate of increase of one of the two variable quantities, while the motion of the disk depends on the rate of increase of the other

* 11th Report of the British Association, p. 308.

† L. E. and D. Phil. Mag., vol. xxx., p. 171.

Figs. 1, 5, 6, and 7.



variable quantity. If now the roller B which is driven without circumferential slipping by the disk has a greater angular velocity than the screw, it will, by its action on the screw, be moved inwards towards the centre of the disk until its rate of motion is the same as that of the screw. If its rate of angular motion is less it will be moved by the screw outwards until in a similar way it is at rest. This may be simply expressed in symbols thus—

Let

ω=angular velocity of screw EE, *i.e.*, of roller B.

ω_1=angular velocity of shaft F, *i.e.*, of roller A.

R=radius of disk.

y=distance of B from centre of disk.

Then if, as is assumed to be the case, diameter of A=diameter of B

$$\frac{\text{angular velocity of B}}{\text{angular velocity of A}}=\frac{\omega}{\omega_1}=\frac{y}{\text{R}}=y\text{K}.$$

K being constant and equal to $\frac{1}{\text{R}}$.

That is, y is a direct measure of the ratio $\frac{\omega}{\omega_1}$.

Now there is no limit in theory to the speed of the screw (EE) or of the roller (B), consequently the latter may be caused to occupy its correct position with any required rapidity.

Theoretically, therefore, y may be made to measure at every instant the ratio of the angular velocities, *i.e.*

$$y\mathrm{K} = \frac{\omega}{\omega_1} = \frac{\frac{dz}{dt}}{\frac{dx}{dt}} = \frac{dz}{dx}$$

where dz and dx correspond respectively to the rates of change of the two variable quantities at any time t.

The author three years ago applied this principle to the construction of a speed indicator. The disk was driven by a clock, and the screw by the body whose speed was to be indicated. In this case ω_1, the angular velocity of the shaft A, was constant.

Then

$$\omega = y\frac{\omega_1}{\mathrm{R}} = y\mathrm{K}.$$

There were in the above instrument two screws of opposite pitch working independently upon one axis, so that each half of the diameter of the disk, available upon either side of its centre, could be employed. On either screw was a roller, working as a nut and connected with a corresponding index upon a dial, so as to always indicate its position upon the disk. One dial was arranged to indicate the speed of the engines of a steam vessel, while at the same time the other was indicating its corresponding speed through the water, so as to simplify the work of progressive speed trials. Two practical advantages may be noticed in connexion with such an instrument. The first is that a speed indicator of this kind may be placed in any part of the ship and communication made electrically, as was done in the above case. The second is that the slight and irregular variations in velocity which in most speed indicators, such as the strophometer of HEARSON and the tachymeter of BUSS, necessitate special arrangements of springs to diminish the oscillations of the index hand, are not recorded with this kind of instrument. This latter fact is the result of the gradual action of the screw, which may, however, be so arranged as to cause an indication of any required degree of sensitiveness. It is not necessary to further describe the instrument, which suffers from the inherent defects of the disk and roller, and the principle of which the author found had previously been suggested by two correspondents in 'Engineering,'* and very possibly by others. Quite recently (May 24) a speed indicator of this kind was exhibited before the Physical Society by Mr. W. GOLDEN and Sir A. CAMPBELL, in which the disk was replaced by a cone, and on the same occasion Mr. N. BAILEY described a speed indicator of his own which was no other than the simple disk and roller applied in this way.†

Enough has been said to show that a great many attempts have been made to employ for practical purposes the invaluable principle of the disk and roller in two distinct ways, one of which is the converse of the other.‡

* 'Engineering,' vol. xx., p. 314.

† 'Nature,' vol. xxx., p. 140.

‡ There is, it is true, yet a third way of using the disk and roller, which is most important and

It now, therefore, becomes necessary to examine the defects of the disk and roller in view of both the foregoing purposes. These defects may be said to be of two kinds—

(1.) Those which are the result of the principle of action itself.

(2.) The limited range of action of the instrument.

(1.) The very conditions under which the disk and roller works are contradictory. On the one hand the roller must slide sideways, that is, in a perpendicular direction to its plane of rotation, or the relative velocity cannot be changed. On the other hand no sliding or slipping must take place in the direction of its rotation, which must be only a motion of pure rolling contact. The roller has to work upon continuously-changing circles, and nothing in the nature of a toothed or serrated edge is admissible. Such a serrated edge has, indeed, been introduced by some inventors, and the surface of the cone or disk on which it works made, as of course it must be, of softer metal. This was so in the instrument of MOSELEY for integrating the work of a steam engine; but it is a significant fact that the committee speak of the 'slight furrows' caused in consequence upon the driving cone.* Now it is easy to see that these slight furrows must introduce an error, as the position of the roller continually changes, and quite vitiate the differential principle of action. The force of friction which must therefore be employed to ensure rolling contact leads to the three following defects:—

(i.) Grinding action between the edge of the roller and face of the disk.

(ii.) Necessity for the application of force in order to change the position of the roller.

(iii.) Error in numerical results.

(i.) The first of these is well known, and is in fact illustrated in an extreme case by every mortar or pug mill. As the edge, which must initially have *some* appreciable thickness, however slight, grows wider, the evil increases, and the size of the roller altering the accuracy of its record is destroyed. This must take place rapidly in the case of the steam engine integrator, where such a wide range of motion occurs at every stroke of the engine.

(ii.) The second defect is a serious one where the instrument is employed for ergometrical purposes, as even allowing that the friction may be always constant at one portion of the disk, it cannot be so where the conditions of rolling are different.

Where the disk and roller is employed as a screw for the converse process, the same actual side friction must also take place, though it is not so apparent. The objects for which it has been more frequently suggested or employed require the use of a clock, and the author has, in endeavouring to apply the principle to the above-mentioned

entirely distinct in principle from the two already discussed, but its application is only suitable for numerical and *discontinuous* calculation; and as, moreover, it involves no new mechanical arrangement, it is not alluded to until hereafter (p. 385), when discussing the *other* purposes of the new mechanism.

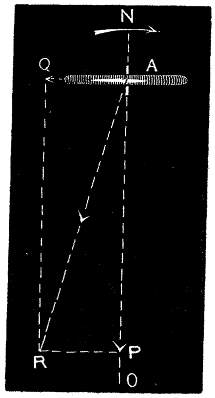
* 11th Report of the British Association, p. 321.

speed indicator, experienced no small difficulty in consequence of the powerful and expensive clockwork required to ensure a uniform speed of rotation.

It is with combinations of this mechanism that the evil results of this loss of power are most evident, not merely because of the limited extent to which it can only in consequence be applied, but because of the unavoidable introduction of errors.

(iii.) The third and last objection, first pointed out by Professor CLERK MAXWELL, is perhaps the most serious of all.

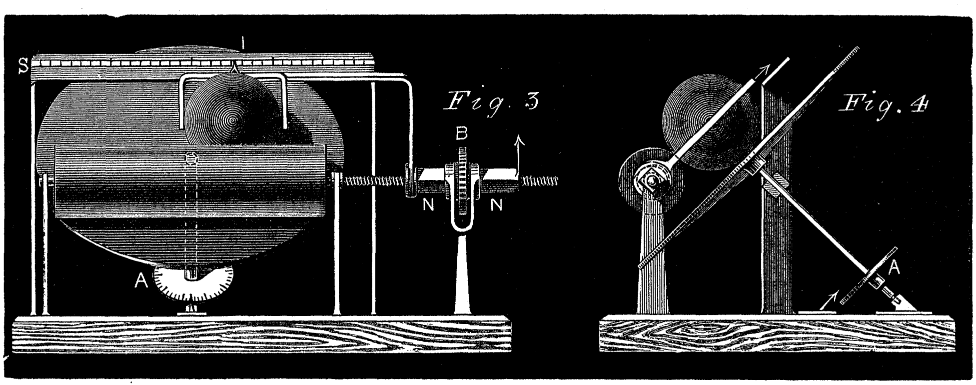
Fig. 2.



Let A (fig. 2) be a plan of the roller, O being the centre of the disk which is turning in the direction indicated by the arrow N. Let A Q be the reaction to the force required to drive the roller round, acting at its periphery at the point of contact with the disk. This force though small must exist, and the result is that the displacement of the roller when the larger force A P (which is that tending to alter its position *radially* on the disk) comes into operation is not radial but along A R. This line A R is the direction of the resultant of A P and A Q, and by moving along it the roller really *slips* through a circumferential distance R P, which represents the *actual error* thus *introduced into the result.* The total error is thus proportional to the distance moved sideways by the roller, that is, to its transverse displacement. It is this fact which is ignored in the tests of ergometers. The common method of procedure is to turn the instrument through a certain distance with a load deflecting the spring to a known and constant amount, and the roller consequently at one radial position on the disk. This is repeated for various other positions of the roller when the record of work done is in each case found to be correct, and it is hard to see why it should be otherwise. These are not, however, the practical conditions of working, which, as a rule, are totally different. In ergometers and steam engine integrators, for instance, a continual change takes place in the position of the roller due to fluctuations of power or steam pressure. This introduces an error in the way described which it does not appear possible to allow for even if the law of variation be known, without elaborate calculation. It certainly would be difficult to test an ergometer or similar instrument with a view to ascertaining and allowing for this error since exactly similar conditions to those occurring in practice must be ensured, and it may be fairly asked, What check is there upon the records of the large number of instruments of this kind in use ?

It was for the purpose of obviating the defects of combined sliding and rolling action* that Professor JAMES THOMSON invented the disk- globe- and cylinder-integrator. This mechanism was at once applied by Sir WILLIAM THOMSON for purposes which had previously been deemed incapable of mechanical treatment, viz., the analysis required in connexion with tide-calculating machines. But, more than this, it was shown by Sir WILLIAM THOMSON that the combination for purposes of integration, now possible from the reduced friction, were not only applicable for calculating $\int\phi(x)\psi(x)dx$, the integral of the product of two functions, but could in theory integrate linear differential equations of any order with variable coefficients.† In one application of the mechanism which has been devised by that gentleman for the solution of a differential equation two machines are employed. The fork of one sphere is connected with the cylinder of the second, and the fork of the second sphere with the cylinder of the first. Thus the motions g_1 and g_2 of the centre of the sphere are obtained, and by eliminating the latter, the former (g_1) is found to be the solution of the differential equation. This method requires the *measurement of the movement* of the fork, that is, of the centre of the sphere, to obtain the result. The use of the *position* of the fork indicated on a scale at every instant in order to obtain a differential coefficient does not appear to have been proposed, and the author therefore ventures to suggest the following arrangement of the disk- globe- and cylinder-mechanism for that purpose.

Figs. 3 and 4.



Figs. 3 and 4 represent this arrangement in front and side elevation. The disk, as in the corresponding disk and roller mechanism, is driven by the wheel A, at a speed which varies as one variable quantity, and communicates a corresponding motion to the globe, and thus to the cylinder. The axis of the cylinder is prolonged, and upon it is cut a screw which works through a nut N. This nut forms the axis of the second

* 'Proceedings of the Royal Society,' vol. 24, p. 263.

† *Ibid.*, pp. 266, 271.

wheel B, which is likewise driven at a speed which varies with the other variable quantity. The wheel B, though not rigidly connected with N, compels it by means of its square or polygonal section to rotate with the same angular velocity, while at the same time allowing it to slide freely in a longitudinal direction. If, now, the wheel A, or what is the same thing, the cylinder, has the same angular motion as the wheel B, then the nut N remains at rest. If the motion is not the same, the nut is moved by the action of the screw until it is so. The position of the centre of the globe, or of the index I on the scale S, indicates at a glance the ratio of the increments of the two quantities.

Just as in the case of the disk and roller, by causing the motion of the wheels A and B to be sufficiently rapid, the differential coefficient may be approached with any required degree of closeness according to the equation

$$yK=\frac{\omega}{\omega_1}=\frac{dz}{dx},$$

where y is the distance of the centre of the sphere from the centre of the disk. If this mechanism were applied to measure velocity, the wheel A would be driven directly by the clock, the wheel B by a machine whose velocity is to be measured.

Another mode of dealing with the resistance of the roller to sliding has been suggested by Mr. VERNON BOYS,* in connexion with a different mechanism. This method consists of a very ingenious device which the inventor terms a "mechanical smoke ring," but though described by means of a detailed drawing it does not appear to have been actually constructed, and indeed it is not easy to see how this could be accomplished to ensure a piece of mechanism giving accurate results.

(2.) It is now necessary to consider the second kind of objection, viz., the limited range of measurement of the disk and roller. This does not affect the magnitude of the growth of either variable, but it does affect the measurement of their ratio, which is given by the radius of the circle on which the roller or sphere is turning on the disk. In theory it is only necessary to alter the units and reduce the scale to any required extent. This, in the integrating form, merely results in magnifying the errors near the zero position, that is near the centre of the disk. In the converse application it has a more serious result, which must be briefly considered.

Bearing in mind that it is the *ratio* of the rate of change of the two quantities which is being now considered, let it be assumed that this with each of them in turn becomes relatively very small. 1st. Let this take place with the one which regulates the motion of the screw; in which case the roller B, fig. 1, simply moves inwards towards the centre of the disk, and will register to the limiting value however small. That is in the notation already adopted, since

$$\omega=y\frac{\omega_1}{R}$$

* 'Philosophical Magazine,' vol. xxii., p. 80.

In the limit when

$$\omega=0 \quad y=0$$

2ndly. If the rate of change of the *other* quantity, that is, the one connected with the disk, becomes very small, then the roller B moves *outwards* and reaches the edge of the disk, at which point

$$y=\mathrm{R} \text{ and } \omega=\omega_1$$

that is, with the equal rollers (A and B) it is impossible to indicate a ratio less than unity. It should be noted that any increase, however great, of the quantity now being considered can be (in theory) measured, for if ω_1 becomes very great, y becomes very small, and may approach as nearly as desired the limiting case when

$$y=0 \quad \omega_1=\infty.$$

Any advantage arising from this latter consideration does not counterbalance the disadvantage that when $\omega=\omega_1$ any further decrease in the speed of the roller A causes B to leave the disk and necessitates special arrangements, not so simple as it might at first be supposed, to bring B on the disk again.

In the cases in which this kind of mechanism would most probably be applied in practice, a clock would be used, so as to make one of the two speeds constant, and only introduce one variable quantity. It is evident from what has been said that the clock would always be employed to drive the disk by means of the roller A, for then the maximum rate change of the variable, which variable might for instance be velocity, would be previously ascertained. The dimensions could then be arranged so that the indicating roller B would never leave the disk, while the lowest velocities, down to the stopping of the body in motion, would be recorded. The author found that in designing the speed indicator previously alluded to, a suitable velocity of the disk was not easy to attain without unduly increasing its size and introducing consequent mechanical disadvantages.

In order to obtain unlimited range within a small compass, the author therefore designed the arrangement shown in fig. 5. The driving roller A′ is directly connected with the other roller B′, so that their radial positions on the disk change together and to an equal extent, their distance apart being equal to the radius of the disk (R). At the same time their angular motions are independent of each other.

Let z=distance of A′ from centre of disk, then

$$\frac{\omega}{\omega_1}=\frac{y}{z}$$

but

$$z=\mathrm{R}-y$$

therefore

$$\omega=\frac{y}{\mathrm{R}-y}\omega_1$$

Let

$$y=0 \quad \text{then } \omega=0$$
$$y=z=\mathrm{R}-y \quad ,, \quad \omega=\omega_1$$
$$y=\mathrm{R} \quad ,, \quad \omega=\infty \text{ (i.e., may approach that value).}$$

Though the greatest possible range is thus obtained the result is not really so convenient for practical purposes. For instance, with an ergometer or steam engine integrator one advantage of the former mechanism is lost, for now work is not registered in simple proportion to the deflection of the spring. For the converse purposes the scale would merely have to be graduated according to the equation betweed ω and y (ω_1 being constant).

There is, however, a connexion between the two forms of disk and roller which bears a resemblance of considerable interest to two corresponding forms of sphere and roller, and when followed out leads to important results.

In fig. 6 let O be the centre of a sphere which rotates on two fixed centres C and C_1 that is about the axis C C_1.

Let a be the point of contact of the roller A which works against, or rolls upon, the great circle of revolution (*i.e.*, the equator), and b the point of contact of the roller B which rolls on a small circle whose radius is bq.

Let angle of plane of rotation of B with axis C $C_1=\alpha$
then

$$\frac{\text{angular velocity of B}}{\text{angular velocity of A}}=\frac{\omega}{\omega_1}=\frac{bq}{\mathrm{O}a}$$

but

$$\frac{bq}{\mathrm{O}a}=\frac{bq}{\mathrm{O}b}=\sin\alpha$$

therefore

$$\frac{\omega}{\omega_1}=\sin\alpha$$

or

$$\omega=\omega_1\sin\alpha$$

In fig. 7 the roller A′ is movable as well as roller B′. They are both attached to the same movable frame—their planes of rotation being always perpendicular to each other.

Then

$$\frac{\text{angular velocity of B}'}{\text{angular velocity of A}'}=\frac{\omega'}{\omega_1{}'}=\frac{b'q'}{a'p'}$$

but by similar triangles $\mathrm{O}'q'b'$, $\mathrm{O}'p'a'$

$$a'p'=\mathrm{O}'q'$$

therefore

$$\frac{\omega}{\omega_1}=\frac{b'q'}{O'q'}=\tan\alpha$$

or

$$\omega=\omega_1\tan\alpha$$

These may be called the "sphere and roller" arrangements or mechanisms. The relation between the two forms of disk and roller is now clear. The first is derived from the more general sine or cosine (or secant, or cosecant) form of sphere and roller mechanism, the second may be in the limiting positions compared with the tangent or cotangent form.

For in the first case, when

$$\alpha=0 \quad \text{then } \omega=0$$
$$\alpha=90^\circ \quad ,, \quad \omega=\omega_1$$

or as before by changing the rollers

$$\alpha=90^\circ \quad \text{then } \omega=\omega_1$$
$$\alpha=0 \quad ,, \quad \omega_1=\infty$$

So that the range in the sine form is either from 0 to ω, *or* from ω_1 to ∞.

In the second case, when

$$\alpha=0 \qquad \omega=0$$
$$\alpha=45 \qquad \omega=\omega_1$$
$$\alpha=90 \qquad \omega=\infty$$

So that the range in the tangent form is (theoretically) from 0 to ∞.

These results were brought by the author before the Physical section of the Bristol Naturalist Society in November last, and illustrated by a model with a wooden sphere 6″ in diameter. It was, however, in endeavouring to apply them to practical purposes that the author was led to the investigation which has resulted in the present communication.

In fig. 6 take any point P in the axis of rotation of the sphere. Draw P F in a direction perpendicular to the plane of rotation of the roller B. Then by a suitable mechanical device consisting of a cross F, sliding upon a rod O T through which another rod turning about the centre P freely passes, it is evident that for any value of the angle α

$$\frac{PF}{OP}=\sin\alpha$$

but by previous reasoning it may be arranged that

$$\frac{\omega}{\omega_1}=\frac{dz}{dx}=\sin\alpha$$

also

$$OP = \text{a constant} = k$$

therefore

$$PF = k\frac{\omega}{\omega_1} = k\frac{dz}{dx} = y$$

So that P F is a direct measure of the ratio, as is the radial distance of the roller in the simple form of disk and roller. Moreover if $s =$ distance turned through by a point in the circumference of B in any time

$$s = b\int_0^s dz = \frac{b}{k}\int_0^s y\,dx$$

where b is the radius of roller B.

In fig. 7 let P′ be again the fixed point, but now take P′ F′ perpendicular to the axis of rotation of the sphere instead of as previously to the plane of roller B, and fix it rigidly in this position.

Employ at F′ a sliding *swivel* instead of a cross as before, through which O′ F′ freely passes, so that as the angle α' changes in any way

$$\frac{P'F'}{O'P'} = \tan\alpha'$$

or as before from previous reasoning

$$\frac{P'F'}{O'P'} = \tan\alpha' = \frac{\omega}{\omega_1} = \frac{dz}{dx} = \frac{y}{k'}$$

or

$$P'F' = k'\frac{dz}{dx} = y$$

or

$$s' = b\int_0^{s'} dz = \frac{b}{k'}\int_0^{s'} y\,dx$$

The sphere and roller mechanism might, therefore, be at once employed to replace the disk and roller, but with the following important difference. The sine form has still the limited range of the corresponding form of disk and roller ; but the tangent form, while having unlimited range, has not now the inconvenient relation between the variables of the corresponding form of disk and roller, but is as *simple to graduate and read as the other.*

For practical purposes the graduation along the fixed perpendicular bar of the tangent form will be shown to be much more convenient to read than those along the oscillating or swinging bar of the sine form, and therefore would probably be generally employed. For the present in what follows that form will be the only one treated of.

Although one of the practical objections, viz., that of grinding, has been obviated, there is still left that of side slipping of the roller on the sphere when change of velocity ratio is required to take place. This may be obviated in the following manner.

Suppose the fixed centres C and C_1 (figs. 6 and 7), each in contact always with the same point on the spherical surface to be replaced by rollers having horizontal planes of rotation.

It will be seen that as these rollers offer less resistance than the side friction of the other rollers, the sphere will be carried bodily round when the angle α' is changed, and the resistance will now only be that of the turning of the rollers C and C_1. To make the arrangement a practical one, and to hold the sphere in its place, idle rollers must be placed respectively opposite to A and B, and fixed to the same frame in order to counteract their pressure. Also a supporting roller must be placed underneath the sphere, with its plane of rotation always perpendicular to the axis of rotation (C C) of the sphere, or what is the same thing, its axis must be always parallel to that of the sphere, and consequently carried in the same frame as the centres.

It would not be convenient to actuate the rollers A′ and B′, or to employ the screw axis, if the frame carrying A′ and B′ actually moved in position, and there is no reason why this should be done.

There are clearly two distinct frames, and two only to be dealt with, one carrying the centres and supporting roller, the other carrying the two rollers A and B, and the idle ones opposite to them. The motion of the two frames being purely relative may be reversed, and the frame carrying the centres made the movable one.

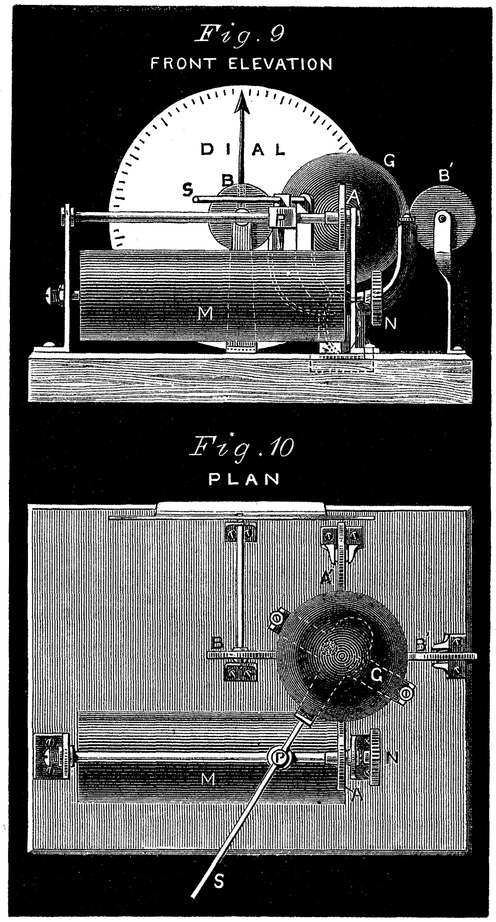
Thus the sphere will now rotate about *movable centres.*

To put these ideas into practical shape, and to see if the rolling centres would answer, a model was made to integrate areas. This is shown in plan, and front elevation, in figs. 9 and 10. A sphere of boxwood (G) is held in a frame simply formed by bending round suitably shaped sheet brass, which thus caused the rollers to grip the sphere. The rollers, A A′ B B′, were disks of boxwood, and merely had common red elastic bands cemented upon their peripheries. The paper on which is the area to be integrated is folded round a cylinder (M), and held there by two india-rubber bands. The cylinder is turned by means of a milled wheel, N, with one hand, and the pointer P, which is connected with the movable frame of centres, is kept on the curve with the other. The roller A works in contact with the surface of the paper, and communicates its motion by frictional contact of the indiarubber to the sphere. The motion of the pointer connected with the axis of B is registered on a suitable dial.

The model not only worked entirely in obedience to the movable centres as far as its range of action permitted, and required only the application of an extremely small force to change the relative position of the frames (*i.e.*, the velocity ratio of A and B), but although only roughly made gave approximately correct results. It was found, however, that the weight of the sphere was not sufficient to keep it accurately in its place, and the more elaborate integrating machine shown in plan and elevation (figs. 11 and 12) has been constructed.

In this latter instrument rollers are placed both above and below the sphere, the distribution of all the rollers being shown by figs. 13 and 14 (p. 17), which are respectively a diagrammatic plan and elevation of the arrangement. In fig. 11 the movable frame is for the sake of clearness omitted, but it is shown in plan in fig. 12.

Figs. 9 and 10.



The principle upon which the area of any curve is integrated when the pointer has passed round the periphery is perfectly simple. In fig. 15 (p. 383) let ON, the axis of x, be the trace of roller A on the cylinder, let P be the position of the pointer at any point on the curve for which the ordinate

$$\mathrm{PP'}=y$$
$$\mathrm{PN}=y_2$$
$$\mathrm{P'N}=y_1$$

The motion of the roller B when the pointer passes along the line $h\mathrm{P}g$ gives in suitable units

$$\int y_2dx = \text{area of curve } h\mathrm{P}glm.$$

The motion of B when the pointer passes along $g\mathrm{P}'h$ is

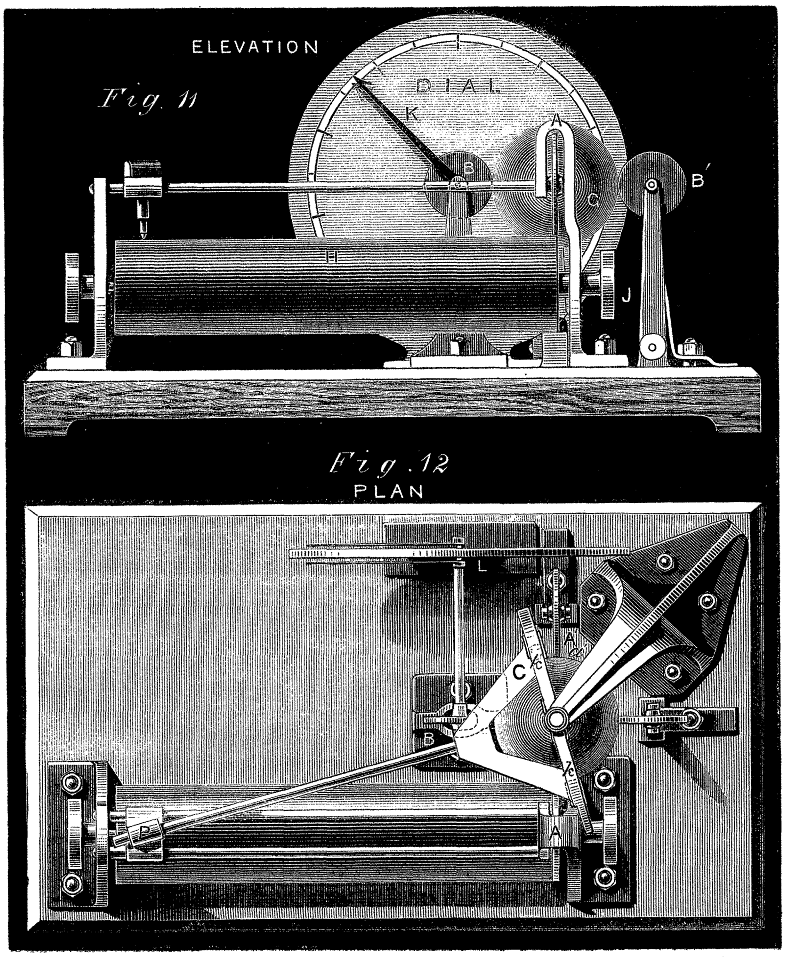
$$-\int y_1 dx = \text{area of curve } g\mathrm{P}'hml$$

because the roller A is moving in the opposite direction, and dx is negative.

Finally, the result shown by the index is

$$\int y_2 dx - \int y_1 dx = \int (y_2 - y_1) dx$$
$$= \int \mathrm{PP}' dx = \int y dx$$
$$= \text{area } \mathrm{P}g\mathrm{P}'h$$

Figs. 11 and 12.



The same result might have been obtained with the sine arrangement (fig. 16), but now either the top or bottom roller in the movable frame must travel along the line of ordinates, and the point N be the centre of rotation for the end of the drum on which the area is wrapped. This is, of course, not so convenient, and its possibility is chiefly interesting from a theoretical point of view.

The satisfactory working of the first model proved that the principle of action was practicable, and some of the most important results were at once evident; one of these being the formation of a chain of such mechanisms in which the loss from friction would be inappreciable; another being the application of a rapidly moving screw with clock-work for such instruments as speed indicators. Moreover, the compound arrangement might be of very compact form. There was one objection which, though of no importance in most applications for purposes of integration, was a serious one for certain applications of the converse process. This was the fact that the movable centres, notwithstanding the great range of velocity ratio, could never take such a position as to give the limiting values in either direction. This is shown at once by fig. 12, where it is seen that to do this, that is, for α to become 0° or 90° the movable centres would have to come into contact with the other rollers A A′ or B B′. It must not be overlooked that although the roller centres are nominally in contact at a point, yet that really the sphere, in turning, twists upon the movable centres at its equator, while motion of the movable frame causes the supporting rollers to twist upon the sphere at its poles. The result is that a smooth, hard, and consequently expensive sphere is required, which in the integrating machine shown in figs. 11 and 12 is made of ivory. It should be noted that even if slight wear takes place at the centres it is distributed over the whole spherical surface, for directly the frame moves round, the former centre becomes a point which, by a sort of precessional action, is not likely to again become the centre, at any rate for a considerable period of time.

It was in endeavouring to overcome these difficulties, and also account for the satisfactory action of the rolling centres, that the author discovered that the foregoing arrangement of movable centres was only a special case of a far more general principle, which, like it, might, with suitable mechanism, be applied to many bodies but which, in the case of the sphere, leads to very practical results.

Let fig. 17 be the perspective view of a sphere. Suppose A C A′ C′ to be the great circle formed by the intersection of a horizontal plane with the spherical surface. Let D C′ D′ C be the great circle formed by a vertical plane. Then these two planes always intersect in a diameter C C′, which may have any direction in the horizontal plane. Suppose a series of rollers, whose planes of rotation are all perpendicular to the horizontal plane, and which are in contact with the great circle A C A′ C′, suppose also a second set of rollers, whose planes are perpendicular to the vertical plane D C D′ C′, and which are in contact with the circle D C D′ C′. Then, by a well-known principle of mechanism, rolling contact can only take place between the spherical surface and the former set of rollers, when the axis of rotation of the sphere is in the horizontal plane, for only then will all their axes intersect the axis of rotation of the sphere. Similarly the rollers round the vertical great circle can only roll on the sphere when its axis of rotation is in the plane of that great circle. *The only axis fulfilling both these conditions is that coinciding with the intersection of the two diametral planes.*

Figs. 13, 14, 15, 16, 18, and 19.

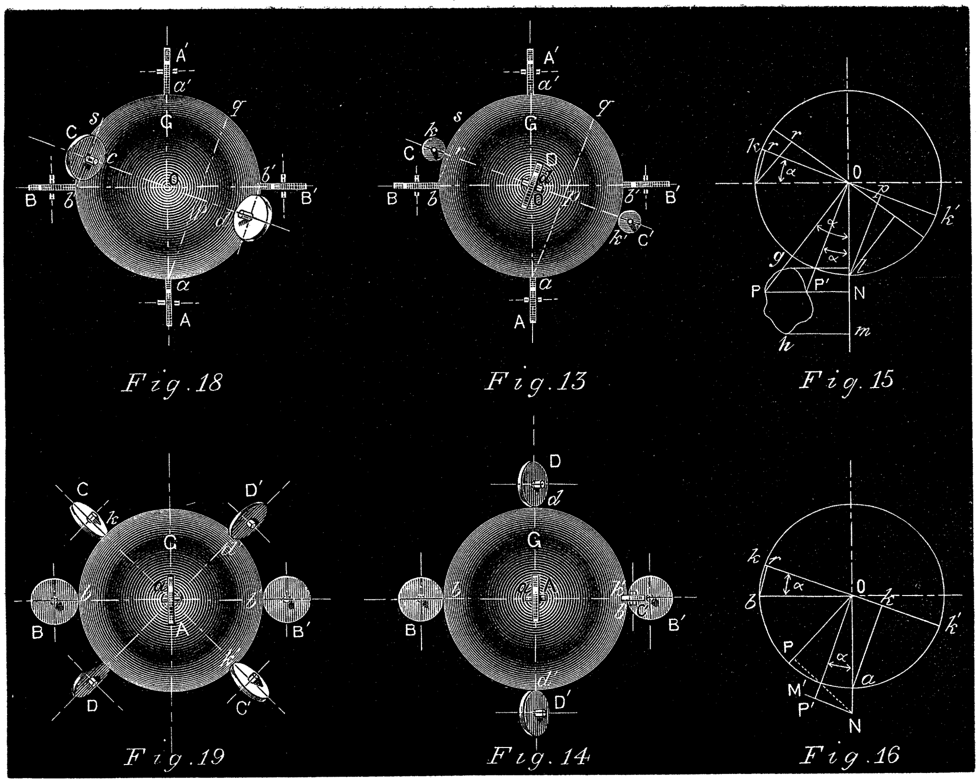
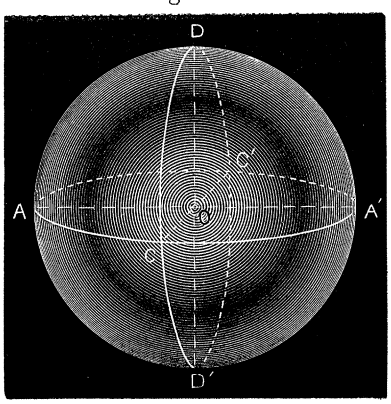


Fig. 17.



It was at the extremity of this axis C C′ (fig. 17) that the movable centres were always in contact; but their successful action was not due to this fact, but simply because they, together with the supporting roller, *fulfilled the other conditions.* These conditions, which can really be approximately, but never perfectly, fulfilled, may be briefly stated as three in number :—

(1.) No slipping must take place.

(2.) Each set of rollers must always be in contact round the great circles formed by two perpendicular (not necessarily horizontal and vertical) diametral planes.

(3.) The axes of all the rollers must intersect the line of intersection of the two planes.

It will be observed that nothing is said about the planes of rotation of the rollers containing the centre of the sphere in the last condition, and the fact that this is not necessary is taken advantage of in the last application which is described in this paper.

From the foregoing results it is evident that it is not the distribution of the rollers round the great circle that is important, but the direction of their axes. Therefore the rollers hitherto alluded to as movable centres do not require to be placed at the points of intersection of the diametral planes, and may be removed to such a position that they can never come into contact with the rollers round the horizontal great circle of contact. At the same time the top and bottom rollers which are not required to be in contact with the poles of the sphere may be removed to any other convenient position along their great circle of contact.

Figs. 18 and 19 (p. 383) show respectively a diagrammatic plan and elevation of the sphere and rollers thus arranged, and the views are lettered to correspond with figs. 11 and 12. Figs. 20 and 21 show the details of the mechanism designed to carry the above principles into operation. The rollers in contact with the horizontal circle are carried on a bracket, which is part of the fixed frame (I I). Those in contact with the vertical circle are carried in a strongly-ribbed movable frame (F F). The mode of attaining numerical results, if required, would be of course identical with that employed in the instruments already described, and would require the additional parts shown in connexion with them.

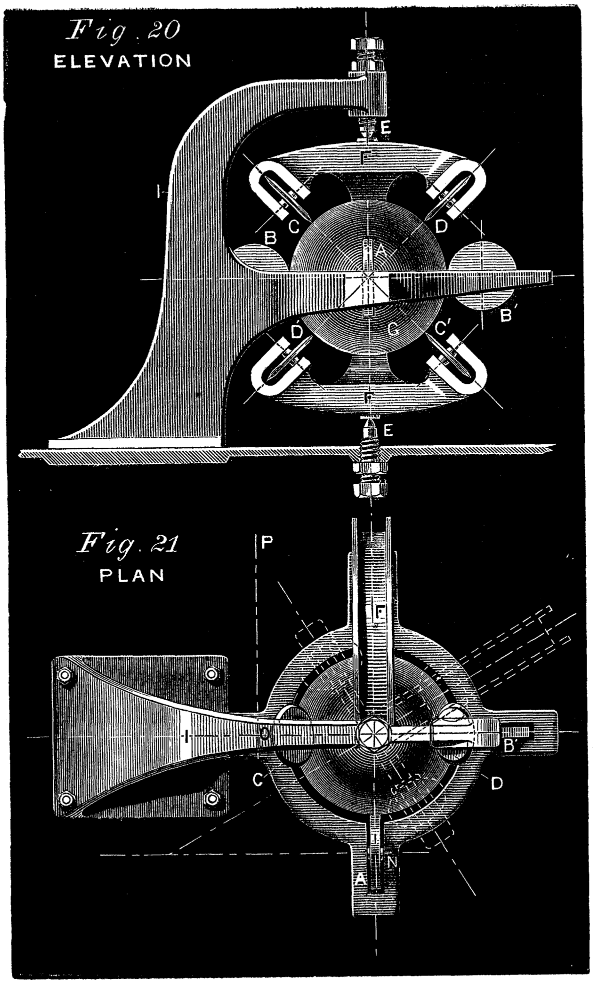
The two kinds of defects shown at the commencement of this investigation to exist in the disk and roller have thus been practically eliminated.

(1.) The results of friction are reduced to a minimum, for not only does the change of velocity ratio, *i.e.*, the relative position of the axes of rotation of the sphere to the fixed rollers, take place by rolling, but the sphere itself is entirely supported by rollers, and thus there is *absolutely nothing but rolling contact.* This was not the case with movable centres. The immediate result of this is that the hard, smooth surface of the sphere is no longer necessary, since no twisting now takes place, and the edges of the rollers transmitting the motion to be measured may now be serrated so as to form the envelopes of their teeth upon the elastic surface of the sphere.

This effects a great reduction of friction, for the pressure which was formerly necessary to ensure the transmission of rolling by frictional contact of the driving roller (A) in the fixed frame is now greatly reduced. The contact is now scarcely frictional contact at all, and the pressure on the bearings is so reduced that the most delicate forces may be dealt with by the mechanism, and accuracy of numerical results ensured.

(2.) With regard to range of action it is clear that theoretically this is now infinitely great in both positive and negative direction, as may be seen by examining the travel of the movable frame (fig. 21).*

Figs. 20 and 21.



Applications for discontinuous calculation.

It has been already shown how, with the sphere and roller, areas may readily be integrated and the operation of differentiation (by employing a screw) be performed. Various other results may however be obtained in a third way which has already been alluded to (p. 371). Before discussing this it may be well to point out that a very simple numerical computator is at once formed by the sphere and roller integrator in the special case where the position of the movable frame is constant during any given motion of the driving roller. A product or quotient of two given numbers may be

* See note at end of the paper.

directly obtained in the following way. Suppose the rollers A and B (fig. 22) to be graduated and a pointer or index attached to each so as to enable the distance turned through by either to be read. Suppose each reading to be brought to zero and the pointer P (connected with the movable frame) moved into a position such that

$$\text{AP} = x = \text{one factor.}$$

Then turn the roller A through a distance such that the index on A reads

$$m = \text{the other factor.}$$

Then by taking suitable units, or what amounts to the same thing, by using suitably proportioned rollers,

$$\text{Reading of B} = n = mx = \text{product of the two given factors.}$$

To obtain a quotient a similar method would be adopted, but the roller B would now be turned instead of A, then

$$\text{Reading of A} = m = \frac{n}{x} = \text{quotient.}$$

The practical application which suggests itself is that of the reduction of tables, in which case one factor is constant. The work might be rapidly performed by always turning A or B (as the case might be) through a constant distance from zero, the pointer P having been first set to the other factor, to an adjustable stop when the reading of the driven roller gives at once the required result.

Thus far the working is only a special case of that of the integrator, viz., where the area is a rectangle, but with the form of instrument for the converse process, that is with the roller A screwed upon its axis, a new principle can be brought into operation. Suppose the roller B (fig. 22) to be turned through a distance n, then the roller A will in consequence turn through a certain distance m, but inasmuch as it forms a nut upon the screw A P, it will at the same time *continually alter* the position of the pointer P, and consequently the value of x, thereby changing the position of the movable centres kk, or axis of rotation of the sphere. Thus the reading (m) of the index of A is a value which is no longer a simple product or quotient but of a nature which must be investigated.

Let θ_θ θ_1 be the angles turned through by A and B respectively.

Let

$l =$ pitch of the screw AP.
$a =$ radius of roller A.
$b =$ " " B.
$x =$ distance AP.
$k =$ " AO.

Then

$$\frac{bd\theta_1}{ad\theta} = \tan\alpha = \frac{x}{k}$$

And

$$x=\frac{l\theta}{2\pi}$$

$$d\theta_1=\frac{al}{2\pi bk}\theta d\theta$$

Then by integration

$$\theta_1=\frac{la}{4\pi kb}\theta^2$$

or

$$\theta=\sqrt{\frac{4\pi kb}{la}}\sqrt{\theta_1}$$

Thus suppose m and n are the respective final readings on the recording dials connected with the rollers A and B, the initial readings having been zero—suppose also that the pitch of the screw is such that the constant quantity $\sqrt{\frac{4\pi kb}{la}}$ is equal to unity.

Then

$$m=\sqrt{n}$$

that is, by turning B through a distance recorded on its own dial as n, the reading (m) on the dial of A gives the square root of the former quantity.

Fig. 22.

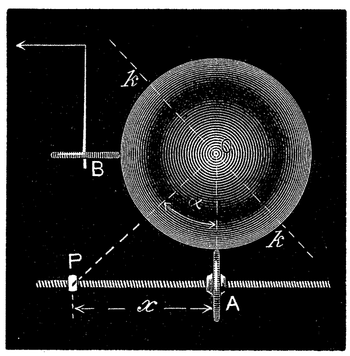
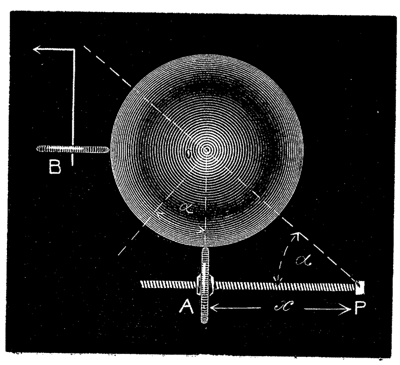


Fig. 23.



Again, suppose P (fig. 23) to be taken on *the other side* of A. Then by similar reasoning since now

$$\frac{bd\theta_1}{ad\theta} = \tan\alpha = \frac{k}{x}$$

$$d\theta_1 = \frac{2\pi ak}{bl} \quad \frac{d\theta}{\theta}$$

and by integration

$$\theta_1 = \frac{2\pi ak}{bl} \log_\epsilon \theta.$$

If, as before, m and n are the respective readings at the end of the operation, and the pitch of the screw be such that

$$\frac{2\pi ak}{bl} = \text{modulus of the Naperian system of logarithms.}$$

Then

$$n = \log_{10} m.$$

In order to graduate the dials in the latter application, since θ can never be zero the limiting position can never be reached; but when

$$m = 1 \quad \text{then } n = 0.$$

therefore the dials of A and B must be adjusted so that these two conditions are simultaneously fulfilled.

In this way it is possible to find the *logarithm* of a number m to any base by merely turning the roller A through that distance and reading the dial of B.

There is another mode of obtaining *any* root, which will be easily understood when it is remembered that by using n sets of spheres and rollers, the value of $(x_1 \times x_2 \times x_3 \times \ldots x_n)$ can be obtained. Make these values of x all equal; then with n frames x^n is given. By turning the last wheel through x^n, and keeping the frames in suitable positions, the first wheel of the series will turn through a distance x.

Thus, if the last wheel be turned through a distance N.

Reading of first wheel or roller

$$= m = \sqrt[n]{\text{N}}.$$

The application of a mechanism to obtain the foregoing results had been previously suggested by Professor MOSELEY in a paper already alluded to (p. 368), but the foregoing investigation was made without any knowledge of this. The modification of Professor JAMES THOMSON's disk- globe- and cylinder-mechanism (figs. 3 and 4) may be at once applied to obtain similar results.

Applications for continuous calculation.

The foregoing applications are for the purposes of obtaining numerical results in which the mechanism is employed, rather as a discontinuous than a continuous calculator. The effects to be obtained by using two or more sets of spheres and rollers in one or both the primary ways, that is for purposes of integration and differentiation, must now be considered.

First let two sets of integrating mechanism be employed. In fig. 24 let

$MN = a$.
$QN = b$.
$\Delta x =$ width of element NQ.
$M =$ moment of area of MN about Ox where Ox is the trace of the roller A on the cylinder.

then

$$M = a\left(\frac{a}{2}+b\right)\Delta x.$$

Fig. 24.

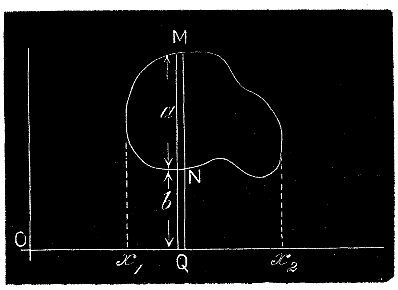
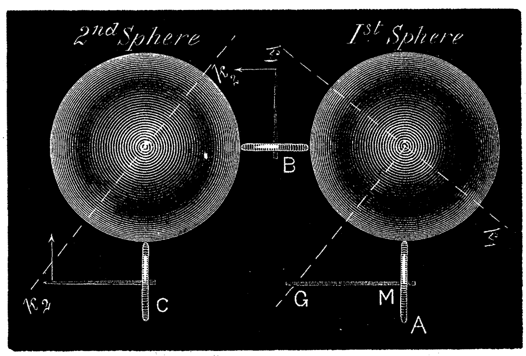


Fig. 25.



Suppose, now, that as the pointer of the integrator travels along the upper portion of the curve, the wheel B, which in the simple integrator records the area, now drives another sphere which has its frame kept so that the value corresponding to y of the original proof is always for that frame equal to $\frac{y}{2}$. This result is easily effected by keeping the frames perpendicular to each other, as in fig. 25, the axes being k_1 k_1 and k_2 k_2 respectively, and using suitable proportions of wheels or rollers.

Then when the roller A travels a small distance Δx at M (fig. 24), the pointer

P describes upon the upper part of the curve say at M, a distance corresponding to upper boundary of element of width Δx. Then:

$$\text{Reading of roller C} = \frac{(a+b)^2}{2}\Delta x.$$

Similarly when the pointer m traverses the lower boundary of the element at N,

$$\text{Reading of roller C} = -\frac{b^2}{2}\Delta x$$

being negative because the roller A is now *returning,* and its motion is reversed.

Taking both results together

$$\text{Reading for element MN} = \frac{(a+b)^2 - b^2}{2}\Delta x$$

$$= a\left(\frac{a}{2}+b\right)\Delta x$$

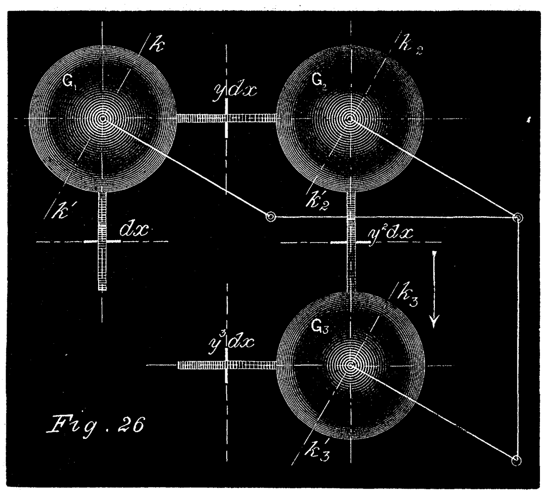
$$= \text{moment of area of element MN about O}x.$$

If now the whole distance moved through round the curve be taken where y_1 and y_2 represent respectively b and $(a+b)$ at any point of the curve, then

$$\text{Reading of C} = \int_{x_1}^{x_2}(y_2{}^2 - y_1{}^2)\,dx$$

$$= \text{moment of area of whole curve about O}x.$$

Fig. 26.



Secondly, let three sets of spheres and rollers be employed, as in fig. 26,*

$$\text{I} = \text{moment of inertia of element MN about O}x_2$$

* The axis $k_2\, k'_2$ should be perpendicular to its present direction in the figure.

then

$$I=\tfrac{1}{12}a^3+a\left(\frac{a}{2}+b\right)^2$$
$$=\tfrac{1}{3}(a^3+3a^2b+3ab^2).$$

Let now the *third* frame, and consequently the axis of rotation of the third sphere, be kept parallel to the first and perpendicular to the second, as in the former case, with the two sets, but let now the units be taken so that the reading of the distance of the pointer P_3 is

$$y_3=\frac{1}{3}y$$

then for the travel of the wheel A over the distance Ox, *i.e.*, the width of the element of area, both forwards and backwards, the reading of the recording wheel D of the third set is

$$=\frac{(a+b)^3}{3}-\frac{b^3}{3}$$
$$=\tfrac{1}{3}(a^3+3a^2b+3ab^2)$$
$=$ *moment of inertia* of element MN about axis Ox (fig. 24).

By taking, as in the previous case,

$$y_1=b$$
$$y_2=(a+b)$$

it is found that final reading of the dial of D for the whole travel of the curve

$$=\int_{x_1}^{x_2}\tfrac{1}{3}\{(y_2-y_1)^3+3(y_2-y_1)^2y_1+3(y_2-y_1)y_1^2\}dx$$
$$=\tfrac{1}{3}\int_{x_1}^{x_2}(y_2^3-y_1^3)dx$$
$=$ moment of inertia of whole area about Ox.

Both the foregoing results may be at once proved in a more general way, thus it is evident that the wheel B is turning at the rate k_1dA where k_1 is some constant, and

A$=$area of curve,

then by means of the addition of the second sphere and roller the product is given as with the first, and C turns at the rate

$$=k_2y(k_1d\text{A})$$

therefore, if

S_2= reading of index for travel of the whole area of curve

$$S_2=k_1k_2\int_{a_1}^{a_2} y dA$$

or with proper units

$$=k_1k_2\int_{x_1}^{x_2}(y_2^2-y_1^2)dx$$

S_2= moment of area about Ox.

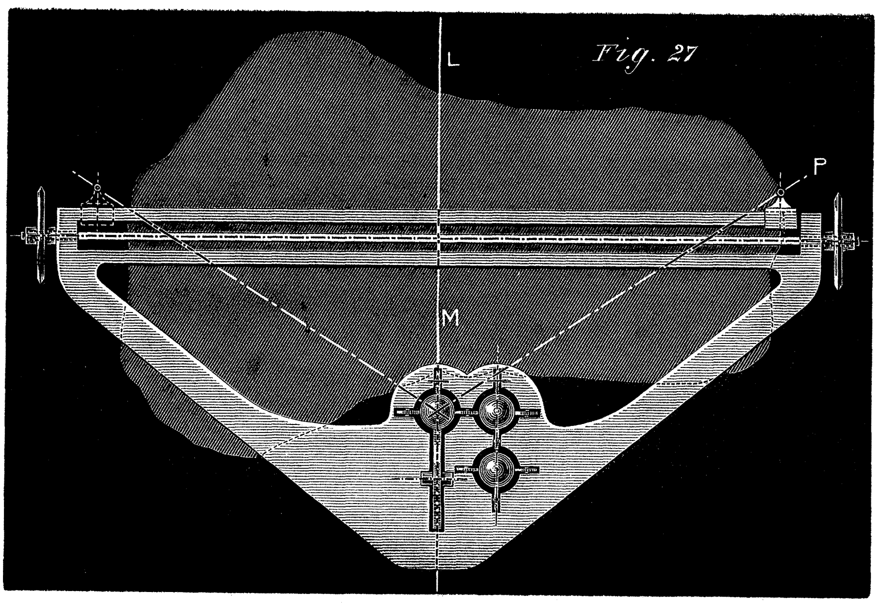
With the addition of the third set of sphere and rollers

$$S_3= \text{reading of index } D=k_1k_2\int_{x_1}^{x_2} k_3y(dM)$$

$$=k_1k_2k_3\int_{x_1}^{x_2}(y_2^3-y_1^3)dx$$

= moment of inertia of area about Ox.

Fig. 27.



The three dials of B, C, and D may be arranged on the parallel ruler as shown in fig. 27, and thus a compact and inexpensive instrument formed, which will give on three dials *simultaneously* and without the *necessity of any calculation whatever,* the three quantities A, M, and I.

But there are certain practical points of particular interest in this instrument which are worthy of consideration. In the first place its symmetrical proportions and wide range of action, as shown by fig. 27, are due to the use of the general form of the sphere and roller mechanism, by which the axis of rotation of the sphere can be made to change across through the centre of the recording roller (B), so as to enable the relative motion of this roller and that of the driving one (A) to be reversed, so that positive and negative values may be obtained.

Suppose the pointer to be on that portion of the curve to the right of the line L M, up which the driving roller is passing (*i.e.*, really the axis of x)—then it has already been proved that the area is integrated by the motion of B. But when the pointer passes the intersection of the line L M with the curve the relative motion of A and B is reversed; this causes B to turn in the opposite direction (unless the widest point of the area is accidentally reached at this point). But the pointer is really now travelling round the opposite way on the curve to the left of L M, and the dial of B is therefore only cutting out or subtracting *the negative portion of the curve.* When the highest point is reached then the motion of the roller A *will be reversed,* and thus along the rest of the curve to the left (since the pointer is still to the left of L M) the motion of the index of B is positive, *i.e.*, the area is *continuously integrated.*

Thus as far as the integration of areas is concerned it is *absolutely immaterial* how the parallel ruler or rolling integrator is applied to the paper, or along what line the roller A runs, whether within or without the curve.

Coming now to the moment of an area it is evident that the position of the line L M has everything to do with the result, but it is clear that mathematically there is an essential difference between this and the first case, for now a negative value of y does not as in the first case give a negative result—from the equation

$$M=\tfrac{1}{2}\int y^2dx$$

and on examination it is found that the index of the wheel or roller C which records this result always moves in *the opposite direction* to the index of B when y is negative. Lastly the equation for moment of inertia

$$I=\tfrac{1}{3}\int y^3dx$$

shows that in this case where y is negative the value of I is recorded in the same way as the area. It will be seen that the mechanism always effects this result, without adjustment or correction by the very principle of its action, and *adds* the moment of inertia for both sides of the line.

This peculiar relation between A, M and I, though evident upon a consideration of what these quantities mean in practical mechanics is thus clearly brought out.

There is no reason (in theory) why the number of spheres should be limited to three or the frames kept parallel or perpendicular to each other. Thus it is possible to obtain the integration of

$$\int F_1(x)F_2(x) \ldots . F_n(x)dx$$

where n is the number of sphere and roller mechanisms, each frame being made to assume a position depending on the respective function with which it deals.

The volumes of solids of revolution can be obtained (with two sets) by merely passing the roller A in direction of the axis of x, and keeping the pointer P on the surface, so that at all times

$$y = \text{radius of circle of revolution.}$$

Then

$$\text{volume} = \pi \int y^2 dx$$

Also with ruled surfaces the pointer of 1st set is kept on one surface, so that

$$y_1 = \text{one ordinate} = y = \phi(x)$$

and

$$y_2 = \text{the other} = z = \psi(x)$$

$$\text{Reading of index of C} = \int yzdx$$

$$= \text{volume} = \int \phi(x)\psi(x)dx$$

Again, in the integration of trigonometrical functions, which are of great importance in naval architecture, it is only necessary to keep the pointer of the index which is attached to the movable frame, and so controls the axis of rotation at the angle, the trigonometrical ratio of which is a function of x, as for instance, in the equation

$$y = \int_0^x \tan \alpha \, dx$$

which gives the area of a curve in which

$$\alpha = \text{angle of heel of vessel}$$

$$x = \text{corresponding ordinate.}$$

All the ratios could be in this way easily dealt with.

The converse application of the sphere and roller mechanism will easily be understood, as the principle has already been fully dealt with in the case of the disk and roller, though only the one example of the speed indicator has yet been suggested—in this case the position of the pointer indicates $v = \frac{ds}{dt}$.

Suppose, however, that a single sphere and roller arranged for the purpose be applied to a single integrating mechanism which is giving the work done in a steam engine as already described (p. 368). The recording wheel of the latter is turning at a rate $=ydx$, where, if W be the measure of the work done,

$$ydx = d\mathrm{W}$$

Therefore if this wheel or roller *actuate* the former single sphere, the pointer of the movable frame of that set of sphere and rollers indicates the result

$$\mathrm{R} = \frac{d\mathrm{W}}{dt} = \textit{rate of working.}$$

In suitable units

$$= \textit{indicated HP.}$$

By causing the pointer in its motion to work the driving wheel of a second sphere whilst a clock drives the screw, the pointer indicates the ratio$=\alpha$.

Where

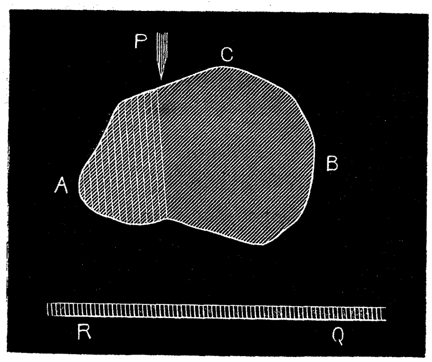
$$\alpha = \frac{d\left(\frac{ds}{dt}\right)}{dt}$$

$$= \frac{d^2s}{dt^2}$$

$$= \text{acceleration, or } \textit{rate of change of velocity.}$$

So in theory any order of differential coefficient $\left(\frac{d^ny}{dx^n}\right)$ may be obtained, n being the number of sets of spheres and rollers used.

Fig. 28.



For integrating the volume of any irregular solid the following approximate method may be suggested. Let A C B, fig. 28, be the section of the solid by the coordinate plane parallel to which the ordinates (y) are measured. Let R Q be a screw the axis of which coincides with the axis of x. Suppose the pointer P to be passed round the solid in planes perpendicular to the above plane and at the same time carried along by the screw R Q—P making one revolution round the solid for one turn of R Q—then the final result of record of the index of roller

$$= \int ydx$$

Let

$$n = \text{number of revolutions of RQ}$$
$$l = \text{pitch of its screw}$$

Then approximately

$$\text{Volume of solid} = nl\int y dx$$

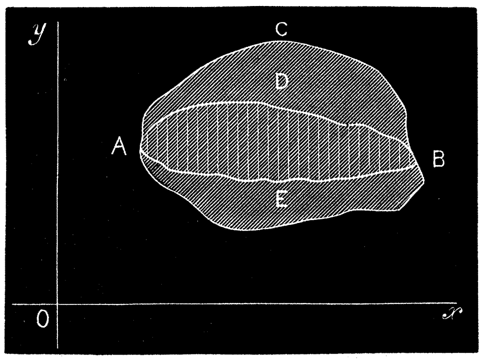
By making l very small the result may be made to approach the true value as nearly as desired.

The operation

$$\iint \phi(x, y) dy dx$$

may, however (in theory), be performed in the following way.

Fig. 29.



Let A C B, fig. 29, be as before the section of any solid. Then to find its volume the foregoing expression would (if possible) first be integrated with respect to y, and an expression of the form Volume$=\int\psi(x)dx$ obtained. The expression $\psi(x)$ may be called the ordinate of *a curve of areas* which may be represented by A D B E, the ordinate D E $=y=\psi(x)$ at any point of which gives the numerical value of the area for that value of the abscissa x. By passing the pointer of the simple integrator over the curve A D B E, the volume of the solid is given at once.

To apply this reasoning to effect a mechanical solution, the pointer of a simple integrator must be passed with sufficient rapidity round the solid as it moves along the axis of x, so that the differential coefficient $\frac{dA}{dt}=$ average areas can be obtained, this gives at any instant the ordinate (D E), which can be used as above stated by means of a suitable mechanism.

Investigation of work lost in friction.

The work lost in friction in the sphere and roller mechanism occurs partly in rolling contact and partly upon the axles, which will be here investigated, is that upon the axles of the movable frame. There are two kinds of effect and two only upon these, viz., direct pressure upon the bearings and twisting of the plane of rotation. The combined effect of these results in

(1.) A direct pressure on the bearings,

(2.) End pressure resulting, according to the construction, in either pivot or collar friction.

Let

$L=$ work lost in one revolution of the sphere
$N=$ number of revolutions of rollers for one revolution of sphere
$W_1=$ pressure on bearings (direct)
$W_2=$ pressure on pivot or collar (end)
$\phi=$ coefficient of friction
$R=$ radius of sphere
$r=$ radius of rollers
$p=$ radius of axles of rollers
$n=$ number of rollers in movable frame.

Then

$$L=L_1+L_2$$

where

$L_1=$ work lost in direct friction by direct pressure on bearing

$$=2\pi\rho NW_1 \sin\phi$$

$L_2=$ work lost in pivot friction

$$=\tfrac{4}{3}\pi\rho NW_2 \tan\phi$$

$$L=2\pi\rho N \sin\phi\left(W_1+\tfrac{2}{3}\frac{W_2}{\cos\phi}\right)$$

Then it is required to find an expression from which N is eliminated and W_1, W_2 expressed in terms of the force to be transmitted.

Let figs. 30 and 31 be respectively a plan and elevation showing diagrammatic views of sphere G with rollers A $A_{,}$ B $B_{,,}$ in contact respectively at points a a' b b'.

Let

P = force to be transmitted acting on periphery of driving roller (generally taken as A in the figures).
Q = reaction of the driver wheel (B).
α = angle of vertical diametral plane to plane of rollers B.
kk' being axis of rotation of the sphere (fig. 30).

Then must

$$Q=P\frac{ap}{bq}=P\cot\alpha$$

otherwise the motion will be accelerated.

Then, since the component of force applied on wheel A normal to the surface, in direction of arrow a', necessary to maintain adhesion, is balanced by the reaction at a, and also that at B by the reaction b', the effect of the two forces P and Q is the same as two forces at the axis of the sphere and two couples about the axis. The forces act respectively at p and q (fig 30). The couples $P\times ap$ and $-Q\times bq$. Their combined effect is

$$M=P.ap-Q.bq=P(ap-\cot\alpha.bq)$$

but

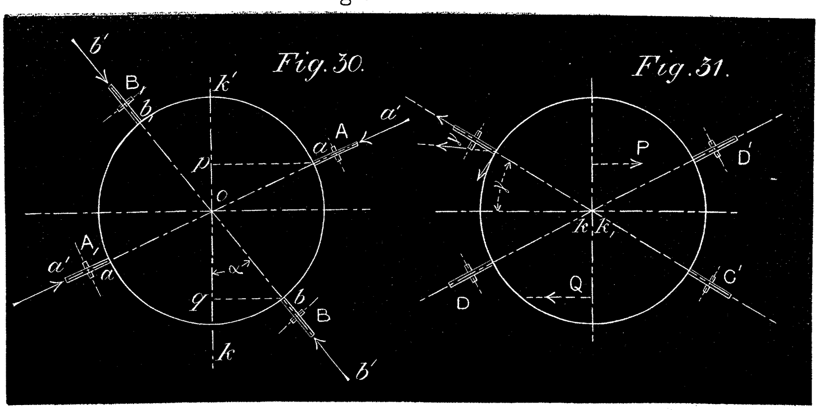
$$ap=R\cos\alpha$$
$$bq=R\sin\alpha$$

Resultant couple about axis

$$=P(R\cos\alpha-R\cot\alpha\sin\alpha)$$
$$=0.$$

Therefore the only effect is that due to parallel forces P and Q acting perpendicular to the axis at the points p and q. The effect of these is equivalent to the couples $(P.op+Q.oq)$ acting in the vertical plane of the axis, and the force $(P-Q)$ acting at the centre of the sphere.

Figs. 30 and 31.



In fig. 31 let

γ = angle of plane of rotation of the rollers with a vertical plane.

Then supposing the rollers to be symmetrically distributed.

(1.) Vertical force on point of contact of each

$$=\frac{P-Q}{2}$$

$$=\frac{P(1-\cot\alpha)}{2}$$

This may be resolved into

$$s=\frac{P(1-\cot\alpha)}{2}\cos\gamma=\text{ direct pressure on bearings}$$

$$r=\frac{P(1-\cot\alpha)}{2}\sin\gamma=\text{ force tending to twist the plane of rotation.}$$

(2.) The couple

$$\begin{aligned}&=P.op+Q.oq\\&=P(op+oq\cot\alpha)\\&=PR(\sin\alpha+\cot\alpha\cos\alpha)\\&=PR\left(\frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha}\right)\\&=\frac{PR}{\sin\alpha}\end{aligned}$$

This acts on the rim of the rollers, and if they were four in number would produce the effect upon each rim

$$\begin{aligned}T_2&=\frac{PR}{\sin\alpha}\times\frac{1}{2R}\\&=\frac{P}{2\sin\alpha}\end{aligned}$$

Summing, now, the total twist

$$T_1+T_2=\frac{P}{2}\left\{\frac{1}{\sin\alpha}+(1-\cot\alpha)\sin\gamma\right\}$$

Let $d=$ half the length of roller axis—that is distance between centres of pressure on either side of roller.

Then the result of this twist is

Direct pressure on pivot or collar

$$=\frac{P}{2}\left\{\frac{1}{\sin\alpha}+(1-\cot\alpha)\sin\gamma\right\}$$

Pressure on bearing

$$=\frac{P}{2}\left\{\frac{1}{\sin\alpha}+(1-\cot\alpha)\sin\gamma\right\}\frac{r}{d}$$

Finally, since

$$N=\frac{2\pi R\cos\gamma}{2\pi r}=\frac{R}{r}\cos\gamma$$

and

$$W_1=\frac{P}{2}\left[\{(1-\cot\alpha)\cos\gamma\}+\left\{\frac{1}{\sin\alpha}+(1-\cot\alpha)\sin\gamma\right\}\frac{r}{d}\right]$$

$$W_2=\frac{P}{2}\left\{\frac{1}{\sin\alpha}+(1-\cot\alpha)\sin\gamma\right\}$$

$$\begin{aligned}L&=2\pi\rho N\sin\phi\left(W_1+\tfrac{2}{3}\frac{W_2}{\cos\phi}\right)\\&=2\pi\rho N\sin\phi\frac{P}{2}\left[\left(\{(1-\cot\alpha)\cos\gamma\}+\left\{\frac{1}{\sin\alpha}+(1-\cot\alpha)\sin\gamma\right\}\frac{r}{d}\right)\right.\\&\qquad\left.+\frac{2}{3\cos\rho}\left(\frac{1}{\sin\alpha}+(1-\cot\alpha)\sin\gamma\right)\right]\end{aligned}$$

Differentiating with respect to γ and equating to zero gives the best angle for γ, therefore

$$\tan\gamma = \frac{r}{d} + \tfrac{2}{3}\cos\phi$$

which result can be at once applied in practice.

In the model made

$$\gamma = 45^\circ.$$

Then

$$\tan\gamma = 1$$
$$\phi = 5^\circ$$
$$\cos\phi = \cdot 996.$$

This gives

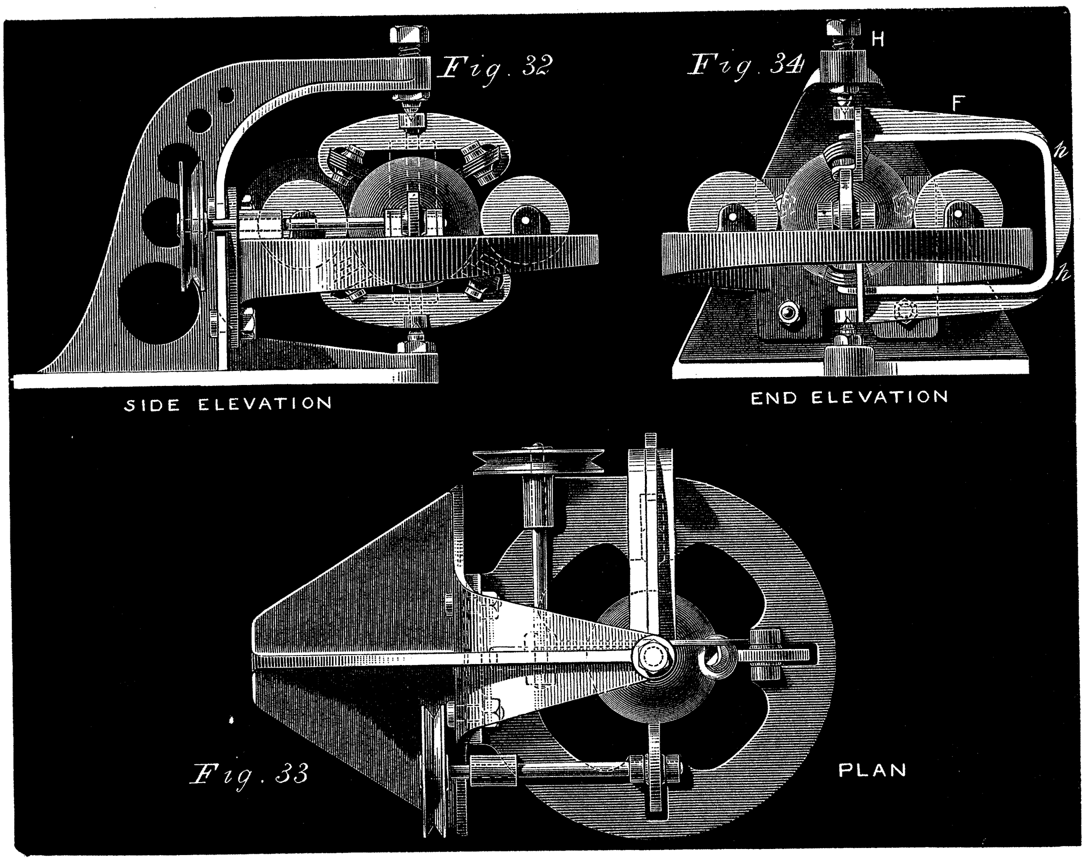
$$\frac{r}{d} = \tfrac{1}{3} \text{ (nearly).}$$

Power Form.

Figs. 32, 33, and 34 show three views of the "power form" of sphere and roller mechanism, which has been constructed to test the actual power transmitted. In this form an important modification, due to the author's brother, Mr. EDWARD SHAW, Whitworth Scholar, Stud. Inst. C.E., has been introduced. It was mentioned that there was no necessity for the planes of rotation of the bearings to contain the centres of the sphere so long as their axes *all* passed through its axis of rotation. It is, therefore, possible instead of merely having sharp-edged disks to act as roller bearings (as in figs. 20 and 21) to have conical rollers. The elastic spherical surface yields to the very slight extent necessary to allow the flat edge of these rollers to form a straight line (instead of a point) of rolling contact passing through the apex of either of the cones, the fustra of which are shown in fig. 32. *Every part of such a cone therefore, rolls upon the sphere.* The action of these cones, which have a number of fine grooves round their peripheries, is even better than was anticipated. There are various points about this design which might be noticed. One is the construction of the frame F (fig. 34), which allows the pressure put upon the centres to close in its two sides and so maintain the requisite pressure on the sphere by means of the locked centre screw (H) without the use of expensive arrangements for springs, which were employed in the movable frame of the second integrating machine. The frame, though ribbed and otherwise rigid, is of smaller cross-section at *h h* (fig. 34) to allow this springing in or out to take place. The convenient nature of the supporting bracket or frame, which is self-contained, and which can be placed anywhere or in any position, is evident, and also the important point of its general simplicity of construction.

It will be seen from the previous investigation that the loss of work in the transmission of power is small and easily calculated, and, moreover, may be reduced by increasing the value of N, and therefore reducing both ρ and P. Practical defects from any possible yielding of the sphere from the effect of the last of these three quantities, viz., P, which cannot well be made a matter of calculation, are thus reduced to any required extent. If the mechanism should prove to be durable, there is, apart from the purely mathematical objects which necessitate accurate working, and which led to its discovery, no apparent limit to its practical applications. It could replace combinations of wheel-work for such purposes as are required in cotton-spinning or

Figs. 32, 33, and 34.



textile machinery, small lathes, electric, and other machines where rapid change of velocity ratio is required, and absolute accuracy is not essential.

The thanks of the author are due to his friends Mr. Edward Buck, M.A., and Mr. C. D. Selman, of whose valuable opinions he has been glad to avail himself in several points in the paper. Also to his brother, Mr. Edward Shaw and to Mr. Charles Bullock, B.A., for the great skill they have shown in constructing the second form of Integrator and the "power" mechanism, and for the assistance of the former in the preparation of the drawings.

Note.

(Added February 4, 1886.)

Some months after the foregoing paper was communicated the author happened to read a letter, written Nov. 5, 1881, by M. Ventosa, of Madrid, and published in 'Nature' (Nov, 24), in which the writer describes a proposed method of obtaining the N., S., E., and W. components of the wind. The method was briefly this. A sphere resting at one pole on a point in the periphery of a roller or disk, and in frictional contact at four points of its equator with four other rollers or disks, would (the planes of all the rollers being vertical) freely revolve upon the turning of the bottom roller. If the plane of the latter were always kept parallel with the direction of the wind, and it were turned at a rate proportional to the wind velocity, and at the same time the planes of the four others were placed in pairs respectively N. and S., E. and W., the latter would record the four corresponding components of the wind. M. Ventosa was led to this idea in 1878, after reading an account of Von Oettingen's Integrating Anemometer, by the endeavour to obviate the sliding friction of that instrument. The arrangement he suggests is *per se* incapable of performing other mathematical operations than simple addition, and, in fact, is only one of the two necessary but distinct features of the author's mechanism mentioned in the abstract of the above paper (Proc. Royal Society, 1884, p. 191). M. Ventosa, however, in his letter, truly says of his proposal, "Cette transmission se fait ici par *roulement sans glissement*" (the italics are his own), and he must be regarded as having, prior to the author, suggested the use of a sphere and rollers for effecting a change in the axis of rotation of the sphere without necessitating other than rolling contact.—H. S. H. S.